Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00071-X
J. Appl. Maths Mechs, Vol. 64, No. 3, pp. 477–483, 2000

0021-8928/00/$—see front matter

AN ELLIPTICAL CRACK UNDER POINT FORCES APPLIED TO ITS SURFACES†

N. M. BORODACHEV

Kiev

(*Received* 16 *June* 1998)

A formula is obtained, by an approximate method, for determining the displacements of the surfaces of an internal elliptical crack in an unbounded elastic solid, when concentrated forces, equal in magnitude and opposite in direction, are applied to its surfaces and the line of action of the forces is perpendicular to the plane of the crack and passes through its centre.

Solutions exist for an internal elliptical crack in the case when the normal and shear stresses, defined by polynomials of arbitrary degree, are specified on its surfaces [1–3]. These solutions were constructed using the properties of the harmonic potentials of an elliptical disc and are characterized by a continuous distribution of the stresses on the crack surfaces.

The solution for an external elliptical crack, under point forces was obtained in [4] using the scantily investigated Lamé functions of the second kind.

1. FORMULATION OF THE PROBLEM

Consider an unbounded linearly elastic solid with a plane internal elliptical crack (a cut). The crack is situated in the $x_3 = 0$ plane, in a rectangular system of coordinates x_1, x_2, x_3, and its centre coincides with the origin of coordinates. The positive orientation S^+ of the crack surface S will be linked to the limiting value $x_3 = 0^+$, while the negative orientation S^- will be linked to $x_3 = 0^-$. Equal but opposite point forces P, the line of action of which coincides with the x_3 axis (see Fig. 1), are applied to the crack surfaces S^+ and S^-.

The integral equation of crack theory in the case of a normal cleavage has the form [5, 6]

$$\sigma_{33}(x_1, x_2) = \frac{\mu}{2\pi(1-\nu)} \nabla^2 \iint\limits_{S^+} \frac{u_3(y_1, y_2)dy_1dy_2}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}}, \quad (x_1, x_2) \in S^+$$
$$\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 \tag{1.1}$$

where u_3 is the displacement of the crack surface, σ_{33} is the normal stress, μ is the shear modulus and ν is Poisson's ratio.

It is assumed in (1.1) that the shear stresses σ_{12} and σ_{13} on the surfaces S^+ and S^- are zero.
The solution of Eq. (1.1) for an internal elliptical crack is well known only when the stress σ_{33} has the form of a polynomial.

We will consider the case when point forces act on the crack, i.e.

$$\sigma_{33}(x_1, x_2) = -P\delta(x_1)\delta(x_2), \quad (x_1, x_2) \in S^+ \tag{1.2}$$

where $\delta(x)$ is the delta function.

We will use the semi-inverse method to solve Eq.(1.1) with condition (1.2). It was used in [7, 8] for an elliptical crack in the case when the stresses on the crack surfaces are expressed by polynomials.

When using the semi-inverse method one must choose for the displacement $u_3(x_1, x_2)$ an expression which satisfies Eq. (1.1) with condition (1.2) and, in the limiting case when $a_1 = a_2$ (a_1 and a_2 are the semiaxes of the boundary ellipse), gives the well-known result for a circular crack.

The solution of Eq. (1.1) with condition (1.2) for a circular crack can be obtained by the method of paired integral equations. As a result we have

†*Prikl. Mat. Mekh.* Vol. 64, No. 3, pp. 497–503, 2000.

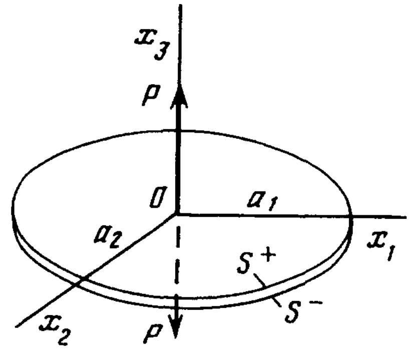

Fig. 1.

$$u_3(r) = \frac{P(1-\nu)}{\pi^2 \mu r} \arccos \frac{r}{a} = \frac{P(1-\nu)}{\pi^2 \mu r} \operatorname{arctg}\left[\frac{a}{r}\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{a^2}\right)^{1/2}\right] \tag{1.3}$$

where a is the radius of the crack.

2. CONSTRUCTION OF THE SOLUTION

In the plane $x_3 = 0$ we introduce the generalized polar coordinates.

$$x_1 = a_1 \rho \cos \phi, \quad x_2 = a_2 \rho \sin \phi \quad (0 \le \rho \le 1, \; 0 \le \varphi \le 2\pi)$$

and, by analogy with formula (1.3), the solution for an elliptical crack will be sought in the form (we assume $a_1 \geqslant a_2$)

$$u_3(\rho, \phi) = \frac{A}{r} \operatorname{arctg}\left[\frac{\beta a_1}{r}(1-\rho^2)^{1/2}\right], \quad \rho \le 1; \quad A = \frac{P(1-\nu)}{\pi^2 \mu}$$
$$r^2 = x_1^2 + x_2^2 = a_1^2 \rho^2 (1 - k^2 \sin^2 \phi), \quad k^2 = 1 - \left(\frac{a_2}{a_1}\right)^2, \quad \rho^2 = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \tag{2.1}$$

where β is a certain constant. The constant A is found from the condition that when $a_1 \to \infty$ and $a_2 \to \infty$ the elastic solid can be split into two half-spaces, unconnected with one another, loaded at the boundary with point forces P, in this case, as is well known [9]

$$u_3 = P(1-\nu)/(2\pi \mu r)$$

We will determine the quantity β. To do this we use the variational formula for a solid with a crack [10, 12]. This formula, in generalized polar coordinates, has the following form for an elliptical crack

$$\delta_n u_3(\rho, \phi) = \frac{\pi(1-\nu)}{2\mu} \int_0^{2\pi} K_1(\varphi; \rho, \phi) K_1(\varphi) \delta n(\varphi) \Pi^{1/2}(\varphi) d\varphi$$
$$\Pi(\varphi) = a_1^2 \sin^2 \varphi + a_2^2 \cos^2 \varphi \tag{2.2}$$

where $\delta n(\varphi)$ is the variation of the crack contour and $\delta_n u_3(\rho, \phi)$ is the variation of the displacement of the crack surface, due to variations of the contour.

Suppose the following condition is satisfied when the boundary contour of an elliptical crack varies

$$\delta a_2 / \delta a_1 = a_2 / a_1 = k_1 \tag{2.3}$$

It can be shown that in case

$$\delta n(\varphi) = a_2 \delta a_1 \Pi^{-1/2}(\varphi)$$

Formula (2.2) then takes the form

$$\delta_n u_3(\rho,\ \phi) = \frac{\pi(1-\nu)a_2\delta a_1}{2\mu} \int\limits_0^{2\pi} K_1(\varphi;\ \rho,\ \phi) K_1(\varphi) d\varphi \tag{2.4}$$

We will use the well-known "test" solution, when a constant pressure, i.e. $p(\rho,\ \phi) = p =$ const, is applied to the surfaces of an elliptical crack. In this case we have

$$u_3(\rho,\ \phi) = \frac{(1-\nu)a_2 p}{\mu \mathbf{E}(k)}(1-\rho^2)^{1/2},\quad \rho \le 1,\quad K_1(\varphi) = \frac{(k_1)^{1/2} p}{\mathbf{E}(k)} \Pi^{1/4}(\varphi) \tag{2.5}$$

where $\mathbf{E}(k)$ is the complete elliptic integral of the second kind.

Using the first relation of (2.5) and taking condition (2.3) into account we obtain $\delta_n u_3(\rho, \varphi)$ and then, from (2.4), we have

$$k_1^{1/2}(1-\rho^2)^{-1/2} = \frac{\pi}{2} a_2 \int\limits_0^{2\pi} K_1(\varphi;\ \rho,\ \phi) \Pi^{1/4}(\varphi) d\varphi,\quad \rho \le 1 \tag{2.6}$$

The quantity $K_1(\varphi;\ \rho,\ \phi)$ is a weighting function for the elliptical crack.

Using (2.1) we determine the quantity $K_1(\varphi;\ 0,\ 0)$, which corresponds to the application of forces $P = 1$ at the centre of the ellipse. We have

$$K_1(\varphi;\ 0,\ 0) = \frac{\mu}{(1-\nu)P} \lim \frac{u_3(\rho,\ \varphi)}{(2r_1)^{1/2}},\quad r_1 \to 0 \tag{2.7}$$

The function $u_3(\rho,\ \varphi)$ is found from (2.1) and r_1 is the distance between the points M and M_1. The point M is situated on the contour of the crack while the point M_1 is close to the boundary of the crack on the inward normal to the crack contour. It can be shown that the coordinates of the point M_1 are

$$x_1 = \cos\varphi(a_1 - a_2 r_1 \Pi^{-1/2}),\quad x_2 = \sin\varphi(a_2 - a_1 r_1 \Pi^{-1/2})$$

where terms of order higher in r_1 are neglected. Consequently

$$(1-\rho^2) = \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right) = \frac{2r_1}{a_1 a_2} \Pi^{1/2}(\varphi) + O(r_1^2) \tag{2.8}$$

Substituting (2.1) into (2.7) and using (2.8) we obtain

$$\begin{aligned} K_1(\varphi;\ 0,\ 0) &= \frac{\beta}{\pi^2 a_1^{3/2}} \Phi(\varphi,\ k) \\ \Phi(\varphi,\ k) &= \left(1 + \frac{k^2}{1-k^2}\sin^2\varphi\right)^{1/4} (1 - k^2 \sin^2\varphi)^{-1} \end{aligned} \tag{2.9}$$

Here we have taken into account the fact that

$$\Pi(\varphi) = a_2^2\left(1 + \frac{k^2}{1-k^2}\sin^2\varphi\right)$$

and the fact that on the contour of the boundary ellipse (when $\rho = 1$)

$$r^2 = a_1^2(1 - k^2 \sin^2 \varphi)$$

Assuming that $\rho = 0$, $\phi = 0$ in (2.6) and substituting $K_1(\varphi; 0, 0)$ from (2.9), we obtain

$$\beta = \frac{\pi}{2(1-k^2)^{1/2} I(k)}, \quad I(k) = \int_0^{\pi/2} \Phi(\varphi, k)\left(1 + \frac{k^2}{1-k^2}\sin^2\varphi\right)^{1/4} d\varphi \tag{2.10}$$

The numerical values of β are given below

k	0	0.1	0.3	0.5	0.6	0.7	0.8	0.9
β	1	0.998	0.976	0.992	0.875	0.806	0.697	0.512

It follows from (2.9) that the stress intensity factor for an elliptical crack of normal cleavage when two point forces P are acting on it (directed along the x_3 axis) will be

$$K_1 = \frac{\beta P}{\pi^2 a_1^{3/2}} \Phi(\varphi, k) \tag{2.11}$$

The numerical values of K_1 calculated from (2.11) and from formula (3.10) of [12], agree, although these formulae were obtained by different methods and have a different form. This, to some extent, confirms the correctness of expression (2.1).

In the limiting case when $a_1 = a_2 = a$ the crack becomes circular and formula (2.1) reduces to (1.3), and from (2.11) we obtain $K_1 = P/(\pi^2 a^{3/2})$, which agrees with the well-known result.

3. GEOMETRICAL INTERPRETATION

We will consider a geometrical interpretation of expression (2.11). For an elliptical crack we have

$$S = \pi a_1 a_2, \quad R = \Pi^{3/2}(\varphi)/(a_1 a_2), \quad ds = \Pi^{1/2}(\varphi) d\varphi \tag{3.1}$$

where S is the area of the ellipse and R is the radius of curvature of its contour. Taking relations (3.1) into account we can write formula (2.11) in the form

$$K_1 = \frac{2}{\pi^{2/3}} \frac{PR^{1/6}}{S^{1/3} I_0 r_0^2}, \quad I_0 = \int_\Gamma \frac{ds}{r_0^2}, \quad r_0 = a_1(1 - k^2 \sin^2 \varphi)^{1/2} \tag{3.2}$$

where Γ is the boundary contour of the crack and r_0 is the distance from the centre of the crack to a point lying on the boundary contour.

To check the correctness of formula (2.1) we will use the reciprocity theorem of [9]

$$\iint_{S^+} \sigma'_{33} u''_3 dS = \iint_{S^+} \sigma''_{33} u'_3 dS \tag{3.3}$$

We will consider the following two states ($(x_1, x_2) \in S^+$)

$$\sigma'_{33}(x_1, x_2) = -p = \text{const} \tag{3.4}$$

$$u'_3(x_1, x_2) = \frac{(1-\nu) a_2 p}{\mu \mathbf{E}(k)} \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)^{1/2}$$

$$u'_3(x_1, x_2) = \frac{(1-\nu) a_2 p}{\mu \mathbf{E}(k)} \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)^{1/2}$$

$$\sigma''_{33}(x_1, x_2) = -P\delta(x_1)\delta(x_2) \tag{3.5}$$

$$u''_3(x_1, x_2) = \frac{P(1-\nu)}{\pi^2 \mu r} \operatorname{arctg}\left[\frac{\beta a_1}{r}\left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)^{1/2}\right]$$

Substituting (3.4) and (3.5) into (3.3) and changing to generalized polar coordinates, we obtain

$$\frac{1}{\mathbf{E}(k)}=\left(\frac{2}{\pi}\right)^2\int_0^{\pi/2}\frac{F(t)dt}{(1-k^2\sin^2 t)^{1/2}} \tag{3.6}$$

where

$$F(t)=\begin{cases}\dfrac{m}{(m^2-1)^{1/2}}\operatorname{arctg}(m^2-1)^{1/2}, & m>1\\ 1, & m=1\\ \dfrac{m}{2(1-m^2)^{1/2}}\ln\dfrac{1+(1-m^2)^{1/2}}{1-(1-m^2)^{1/2}}, & m<1\end{cases}$$

$$m=\beta(1-k^2\sin^2 t)^{-1/2}$$

The integral on the right-hand side of (3.6) was evaluated using Gauss's quadrature formula with 96 nodes. As a result we prove the correctness of Eq. (3.6).

4. A CHECK OF THE CORRECTNESS OF THE SOLUTION

We will now make the main check. We will prove that expression (2.1) satisfies integral equation (1.1) with condition (1.2). i.e. we will prove that the following equality holds

$$\frac{1}{2\pi^3}\nabla^2\iint\limits_{S^+}\frac{1}{r}\operatorname{arctg}\left[\frac{\beta a_1}{r}\left(1-\frac{y_1^2}{a_1^2}-\frac{y_2^2}{a_2^2}\right)^{1/2}\right][(x_1-y_1)^2+(x_2-y_2)^2]^{-1/2}dy_1dy_2= \tag{4.1}$$

$$=-\delta(x_1)\delta(x_2),\quad (x_1,x_2)\in S^+$$

It is obviously impossible to evaluate the integral in Eq. (4.1) in closed form. Using numerical methods it is better to change to elliptic coordinates u, v:

$$x_1=c\operatorname{ch}u\cos v,\quad x_2=c\operatorname{sh}u\sin v\quad (u\geqslant 0, 0\leqslant v\leqslant 2\pi)$$

$$c=(a_1^2-a_2^2)^{1/2},\quad a_1\geqslant a_2$$

In elliptic coordinates Eq. (4.1) takes the form

$$\frac{1}{2\pi^3}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\int_0^{2\pi}dv_1\int_0^{u_0}\frac{1}{r^*R^*}\operatorname{arctg}\left[\frac{\beta}{kr^*}g(u_1,v_1)\right](\operatorname{ch}^2u_1-\cos^2 v_1)du_1= \tag{4.2}$$

$$=-\delta(u)\begin{cases}\delta(v-\pi/2) & (0\leqslant u\leqslant u_0, 0\leqslant v\leqslant\pi)\\ \delta(v-3\pi/2) & (0\leqslant u\leqslant u_0, \pi\leqslant v\leqslant 2\pi)\end{cases}$$

$$r^*=(\operatorname{ch}^2u_1-\sin^2 v_1)^{1/2}$$

$$R^*=[(\operatorname{ch}u\cos v-\operatorname{ch}u_1\cos v_1)^2+(\operatorname{sh}u\sin v-\operatorname{sh}u_1\sin v_1)^2]^{1/2}$$

$$g(u_1,v_1)=\left[1-\left(\frac{k}{k_1}\right)^2(k_1^2\operatorname{ch}^2u_1\cos^2 v_1+\operatorname{sh}^2u_1\sin^2 v_1)\right]^{1/2}$$

$$k_1=a_2/a_1,\quad u_0=\ln\frac{1+k_1}{k}$$

It is inconvenient to apply numerical methods directly to expression (4.2) since it is extremely difficult to obtain the delta function numerically. In view of this we replace (4.2) by the following system

$$f(u,v) = \frac{1}{2\pi^3} \int_0^{2\pi} dv_1 \int_0^{u_0} \frac{1}{r^* R^*} \operatorname{arctg}\left[\frac{\beta}{kr^*} g(u_1, v_1)\right] (\operatorname{ch}^2 u_1 - \cos^2 v_1) du_1 \tag{4.3}$$

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) f(u,v) = -\delta(u) \begin{cases} \delta(v - \pi/2) & \text{при } 0 \leqslant v \leqslant \pi \\ \delta(v - 3\pi/2) & \text{при } \pi \leqslant v \leqslant 2\pi \end{cases} \tag{4.4}$$

In formulae (4.3) and (4.4) $(u, v) \in S^+$, т.с. $0 \leqslant u \leqslant u_0$, $0 \leqslant v \leqslant 2\pi$.

The general solution of Poisson's equation (4.4) can be represented in the form of the sum of some of its particular solutions and functions that are harmonic inside the ellipse. A function that is harmonic inside the ellipse can be represented in the following form [13]

$$F(u,v) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \frac{\operatorname{ch} nu}{\operatorname{ch} nu_0} \cos nv + B_n \frac{\operatorname{sh} nu}{\operatorname{sh} nu_0} \sin nv \right) \tag{4.5}$$

The coefficients A_n and B_n are determined from the limiting (boundary) condition

$$F(u,v)\big|_{u=u_0} = F(u_0, v) \tag{4.6}$$

Consequently,

$$\begin{aligned} A_n &= \frac{1}{\pi} \int_0^{2\pi} F(u_0, v) \cos nv dv \quad (n = 0, 1, 2, \ldots) \\ B_n &= \frac{1}{\pi} \int_0^{2\pi} F(u_0, v) \sin nv dv \quad (n = 1, 2, \ldots) \end{aligned} \tag{4.7}$$

Hence, the general solution of Eq. (4.4) can be represented as follows:

$$f(u,v) = F(u,v) + \frac{1}{2\pi} \ln \frac{1}{[\varphi(u,v)]^{1/2}} \quad (u,v) \in S^+ \tag{4.8}$$

$$\varphi(u,v) = \begin{cases} u^2 + (v - \pi/2)^2 & \text{when } 0 \leqslant v \leqslant \pi \\ u^2 + (v - 3\pi/2)^2 & \text{when } \pi \leqslant v \leqslant 2\pi \end{cases}$$

The function $f(u, v)$ was calculated from formulae (4.3) and (4.8) and the results were compared. The function $f(u, v)$ was calculated at a series of points (u_j, v_j) for different values of the parameter k $(0 \leqslant k \leqslant, 0.9)$. The error in calculating the function $f(u, v)$ using formulae (4.3) and (4.8) did not exceed 0.5%.

The integral (4.3) converges for all points $(u, v) \in S^+$ apart from the point $u - 0$, $v = \pi/2$ (or $v = 3\pi/2$), at which it has a logarithmic singularity.

As extensive numerical calculations showed, expression (4.2) is satisfied with a high degree of accuracy. Hence it follows that expression (2.1) for determining the displacements of the surface of the crack S^+ satisfies Eq. (1.1) with condition (1.2).

REFERENCES

1. KASSIR, M. K. and SIH, G. C., *Three-dimensional Crack Problems*. Leyden: Noordhoff, 1975.
2. BORODACHENV, A. N., Determination of the stress intensity factors for a plane elliptical crack for arbitrary boundary conditions. *Izv. Akad. Nauk SSSR. MTT*, 1981, 2, 63–63.
3. VIJAYAKUMAR K. and ATLURI S. N., An embedded elliptical crack, in an infinite solid, subjected to arbitrary crack-face tractions. *Trans. ASME. J. Appl. Mech.*, 1981, **48**, 1, 88–96.
4. STALLYBRASS, M. P., On the concentrated loading of an external elliptical crack. *Quart J Mech. and Appl. Math*. 1982. V. 35, No. 4. P. 441–459.
5. PARTON, V. Z and MOROZOV, Ye M., *The Mechanics of Elastoplastic Fracture*. Nauka, Moscow, 1985.
6. PANASYUK, V. V., *The Mechanics of Quasielastic Fracture*. Naukova Dumka, Kiev, 1991.
7. SNEDDON, I. N., The stress intensity factor for a flat elliptical crack in an elastic solid under uniform tension. *Intern. J. Engng. Sci.*, 1979, **17**, 2, 185–191.
8. SHIBUYA, T., Some mixed boundary value problems for an infinite solid containing a flat elliptical crack. *Bull. JSME*, 1977, **20**, 146, 909–914.
9. LUR'YE, A. I., *The Theory of Elasticity*. Nauka, Moscow, 1970.

10. RICE, J. R., First-order variation in elastic fields due to variation in location of a planar crack front. *Trans. ASME. J. Appl. Mec.* 1985, **52**, 3,. 571–579.
11. BORODACHEV, N. M., A variational method of solving the three- dimensional problem of the theory of elasticity for a solid with a planar crack. *Prikl. Mekh.*, 1986, **22**, 4, 71–76.
12. BORODACHEV, N. M., A method of constructing the weighting function for a solid with a crack. *Prikl. Mat. Mekh.*, 1998, **62**, 2, 329–333.
13. ANTONOV, V. A., TIMOSHKOVA, Ye. I. and KHOLSHEVNIKOV, K. V., *Introduction to Newtonian Potential Theory*. Nauka, Moscow, 1988.
14. GRADSHTEIN, I. S. and RYZHIK, I. M., *Tables of Integrals, Sums, Series and Products*. Academic Press, New York, 1980.

Translated by R.C.G.